A transfer function characterizes the transient response of a system. In particular, the poles of the transfer function appear in the time-domain representation of the system response. For example, a pole at p results in a term in the response of p^n. Clearly, if the pole has magnitude greater than 1, the response will grow over time. A system is stable if all of its poles are inside of the unit circle, that is, have magnitude less than 1. A system is unstable if at least one of the poles is outside of the unit circle, and a system is marginally stable if it is not unstable and if it has at least one pole on the unit circle.

The transfer function of a system $H(z)$ is related to the frequency response of the system $H(\Omega)$ by the relationship $z = e^{j\Omega}$. In Chapter 5, it was seen that the frequency response is used to determine the response of a system to signals of different frequencies. Moreover, the shape of the frequency response determines the type of filtering present in the system, that is, which frequency ranges are attenuated. An example of a lowpass filter given in this chapter is the IIR EWMA filter.

PROBLEMS

7.1. Consider the discrete-time signal $x[n]$, where

$$x[n] = \begin{cases} b^n, & \text{for } n = 0, 1, 2, \ldots, N-1 \\ 0, & \text{for all other } N \end{cases}$$

Here, b is an arbitrary real number and N is a positive integer.

(a) For what real values of b does $x[n]$ possess a z-transform?

(b) For the values of b determined in part (a), find the z-transform of $x[n]$. Express your answer as a ratio of polynomials.

7.2. A discrete-time signal $x[n]$ has z-transform

$$X(z) = \frac{z}{8z^2 - 2z - 1}$$

Determine the z-transform $V(z)$ of the following signals:

(a) $v[n] = x[n-4]u[n-4]$

(b) $v[n] = x[n+2]u[n+2]$

(c) $v[n] = \cos(2n)x[n]$

(d) $v[n] = e^{3n}x[n]$

(e) $v[n] = n^2x[n]$

(f) $v[n] = x[n] * x[n]$

(g) $v[n] = x[0] + x[1] + x[2] + \cdots + x[n]$

7.3. Compute the z-transform of the discrete-time signals (a)–(j). Express your answer as a ratio of polynomials in z whenever possible. Check your answer by using the Symbolic Math Toolbox command `ztrans`.

(a) $x[n] = \delta[n] + 2\delta[n-1]$

(b) $x[n] = 1$ for $n = 0, 1$ and $x[n] = 2$ for all $n \geq 2$ (i.e., $n = 2, 3, \ldots$)

(c) $x[n] = e^{0.5n}u[n] + u[n-2]$

(d) $x[n] = e^{0.5n}$ for $n = 0, 1$, and $x[n] = 1$ for all $n \geq 2$

(e) $x[n] = \sin(\pi n/2)u[n-2]$

(f) $x[n] = (0.5)^n nu[n]$

(g) $x[n] = u[n] - nu[n-1] + (1/3)^n u[n-2]$

(h) $x[n] = n$ for $n = 0, 1, 2$ and $x[n] = -n$ for all $n \geq 3$
(i) $x[n] = (n-1)u[n] - nu[n-3]$
(j) $x[n] = (0.25)^{-n}u[n-2]$

7.4. Using the transform pairs in Table 7.3 and the properties of the z-transform in Table 7.2, determine the z-transform of the following discrete-time signals:
(a) $x[n] = (\cos^2 \omega n)u[n]$
(b) $x[n] = (\sin^2 \omega n)u[n]$
(c) $x[n] = n(\cos \omega n)u[n]$
(d) $x[n] = n(\sin \omega n)u[n]$
(e) $x[n] = ne^{-bn}(\cos \omega n)u[n]$
(f) $x[n] = ne^{-bn}(\sin \omega n)u[n]$
(g) $x[n] = e^{-bn}(\cos^2 \omega n)u[n]$
(h) $x[n] = e^{-bn}(\sin^2 \omega n)u[n]$

7.5. Let $x[n]$ be a discrete-time signal with $x[n] = 0$ for $n = -1, -2, \ldots$. The signal $x[n]$ is said to be summable if

$$\sum_{n=0}^{\infty} x[n] < \infty$$

If $x[n]$ is summable, the sum x_{sum} of $x[n]$ is defined by

$$x_{\text{sum}} = \sum_{n=0}^{\infty} x[n]$$

Now, suppose that the z-transform $X(z)$ of $x[n]$ can be expressed in the form

$$X(z) = \frac{B(z)}{a_N(z - p_1)(z - p_2)\cdots(z - p_N)}$$

where $B(z)$ is a polynomial in z. By using the final-value theorem, show that, if $|p_i| < 1$ for $i = 1, 2, \ldots, N$, $x[n]$ is summable and

$$x_{\text{sum}} = \lim_{z \to 1} X(z)$$

7.6. Using the results of Problem 7.5, compute x_{sum} for the signals (a)–(d). In each case, assume that $x[n] = 0$ for all $n < 0$.
(a) $x[n] = a^n$, $|a| < 1$
(b) $x[n] = n(a^n)$, $|a| < 1$
(c) $x[n] = a^n \cos \pi n$, $|a| < 1$
(d) $x[n] = a^n \sin(\pi n/2)$, $|a| < 1$

7.7. Let p and c be complex numbers defined in polar coordinates as $p = \sigma e^{j\Omega}$ and $c = |c|e^{j\angle c}$. Prove the following relationship:

$$cp^n + \bar{c}\,\bar{p}^n = 2|c|\sigma^n \cos(\Omega n + \angle c)$$

7.8. A discrete-time signal $x[n]$ has z-transform

$$X(z) = \frac{z + 1}{z(z - 1)}$$

Compute $x[0]$, $x[1]$, and $x[10{,}000]$.

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7.10. For
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7.9. Compute the inverse z-transform $x[n]$ of the transforms (a)–(h). Determine $x[n]$ for all integers $n \geq 0$. Check your answer by using the Symbolic Math Toolbox command `iztrans`.

(a) $X(z) = \dfrac{z}{z^2 + 1}$

(b) $X(z) = \dfrac{z^2}{z^2 + 1}$

(c) $X(z) = \dfrac{1}{z^2 + 1} + \dfrac{1}{z^2 - 1}$

(d) $X(z) = \dfrac{z^2}{z^2 + 1} + \dfrac{z}{z^2 - 1}$

(e) $X(z) = \dfrac{z^2 - 1}{z^2 + 1}$

(f) $X(z) = \dfrac{z + 2}{(z - 1)(z^2 + 1)}$

(g) $X(z) = \dfrac{z^2 + 2}{(z - 1)(z^2 + 1)}$

(h) $X(z) = \ln\left(\dfrac{2z - 1}{2z}\right)$

7.10. For the transforms given in Problem 7.9 (a) to (g), compute the inverse z-transform numerically by using the command `filter`. Compare these results with the answers obtained analytically for $n = 0$ to $n = 5$.

7.11. Find the inverse z-transform $x[n]$ of the transforms that follow. Determine $x[n]$ for all n.

(a) $X(z) = \dfrac{z + 0.3}{z^2 + 0.75z + 0.125}$

(b) $X(z) = \dfrac{5z + 1}{4z^2 + 4z + 1}$

(c) $X(z) = \dfrac{4z + 1}{z^2 - z + 0.5}$

(d) $X(z) = \dfrac{z}{16z^2 + 1}$

(e) $X(z) = \dfrac{2z + 1}{z(10z^2 - z - 2)}$

(f) $X(z) = \dfrac{z + 1}{(z - 0.5)(z^2 - 0.5z + 0.25)}$

(g) $X(z) = \dfrac{z^3 + 1}{(z - 0.5)(z^2 - 0.5z + 0.25)}$

(h) $X(z) = \dfrac{z + 1}{z(z - 0.5)(z^2 - 0.5z + 0.25)}$

7.12. For each of the transforms given in Problem 7.11, compute the inverse z-transform numerically by using the command `filter`. Compare these results with the answers obtained analytically.

7.13. By using the z-transform, compute the convolution $x[n] * v[n]$ for all $n \geq 0$, where

(a) $x[n] = u[n] + 3\delta[n - 1]$, $v[n] = u[n - 2]$

(b) $x[n] = u[n]$, $v[n] = nu[n]$

(c) $x[n] = \sin(\pi n/2)u[n]$, $v[n] = e^{-n}u[n - 2]$

(d) $x[n] = u[n - 1] + \delta[n]$, $v[n] = e^{-n}u[n] - 2e^{-2n}u[n - 2]$

7.14. A linear time-invariant discrete-time system has unit-pulse response

$$h[n] = \begin{cases} \dfrac{1}{n}, & \text{for } n = 1, 2, 3 \\ n - 2, & \text{for } n = 4, 5 \\ 0, & \text{for all other } n \end{cases}$$

Compute the transfer function $H(z)$.

7.15. The input $x[n] = (-1)^n u[n]$ is applied to a linear time-invariant discrete-time system. The resulting output response $y[n]$ with zero initial conditions is given by

$$y[n] = \begin{cases} 0, & \text{for } n < 0 \\ n + 1, & \text{for } n = 0, 1, 2, 3 \\ 0, & \text{for } n \geq 4 \end{cases}$$

Determine the transfer function $H(z)$ of the system.

7.16. For the system defined in Problem 7.15, compute the output response $y[n]$ resulting from the input $x[n] = (1/n)(u[n-1] - u[n-3])$ with zero initial conditions.

7.17. A system is described by the difference equation

$$y[n] + 0.7y[n-1] = u[n]; \quad y[-1] = 1$$

(a) Find an analytical expression for $y[n]$.
(b) Verify your result by simulating the system by using MATLAB.

7.18. Repeat Problem 7.17 for the system described by the following difference equation:

$$y[n] - 0.2y[n-1] - 0.8y[n-2] = 0; \quad y[-1] = 1, y[-2] = 1$$

7.19. A linear time-invariant discrete-time system is described by the input/output difference equation

$$y[n+2] + y[n] = 2x[n+1] - x[n]$$

(a) Compute the unit-pulse response $h[n]$.
(b) Compute the step response $y[n]$.
(c) Compute $y[n]$ for all $n \geq 0$ when $x[n] = 2^n u[n]$ with $y[-1] = 3$ and $y[-2] = 2$.
(d) An input $x[n]$ with $x[-2] = x[-1] = 0$ produces the output response $y[n] = (\sin \pi n)u[n]$ with zero initial conditions. Determine $x[n]$.
(e) An input $x[n]$ with $x[-2] = x[-1] = 0$ produces the output response $y[n] = \delta[n-1]$. Compute $x[n]$.
(f) Verify the results of parts (a) to (e) via computer simulation.

7.20. A linear time-invariant discrete-time system is given by the input/output difference equation

$$y[n] + y[n-1] - 2y[n-2] = 2x[n] - x[n-1]$$

Find an input $x[n]$ with $x[n] = 0$ for $n < 0$ that gives the output response $y[n] = 2(u[n] - u[n-3])$ with initial conditions $y[-2] = 2, y[-1] = 0$.

7.21. The input $x[n] = u[n] - 2u[n-2] + u[n-4]$ is applied to a linear time-invariant discrete-time system. The resulting response with zero initial conditions is $y[n] = nu[n] - nu[n-4]$. Compute the transfer function $H(z)$.

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7.22. A system has the transfer function

$$H(z) = \frac{-0.4z^{-1} - 0.5z^{-2}}{(1 - 0.5z^{-1})(1 - 0.8z^{-1})}$$

(a) Compute an analytical expression for the step response.
(b) Verify your result by simulating the step response by using MATLAB.

7.23. Repeat Problem 7.22 for the system with the transfer function

$$H(z) = \frac{z^2 - 0.1}{z^2 - 0.6484z + 0.36}$$

7.24. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{z^2 - z - 2}{z^2 + 1.5z - 1}$$

(a) Compute the unit-pulse response $h[n]$ for all $n \geq 0$.
(b) Compute the step response $y[n]$ for all $n \geq 0$.
(c) Compute the output values $y[0]$, $y[1]$, $y[2]$ resulting from the input $x[n] = 2^n \sin(\pi n/4) + \tan(\pi n/3)$, $n = 0, 1, 2, \ldots$, with zero initial conditions.
(d) If possible, find an input $x[n]$ with $x[n] = 0$ for all $n < 0$ such that the output response $y[n]$ resulting from $x[n]$ is given by $y[0] = 2$, $y[1] = -3$, and $y[n] = 0$ for all $n \geq 2$. Assume that the initial conditions are equal to zero.
(e) Verify the results of parts (a) to (d) by computer simulation.

7.25. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{z}{(z - 0.5)^2(z^2 + 0.25)}$$

(a) Find the unit-pulse response $h[n]$ for all $n \geq 0$.
(b) Simulate the unit-pulse response by using MATLAB, and compare this result with the result for $h[n]$ obtained analytically in part (a).

7.26. The input $x[n] = (0.5)^n u[n]$ is applied to a linear time-invariant discrete-time system with the initial conditions $y[-1] = 8$ and $y[-2] = 4$. The resulting output response is

$$y[n] = 4(0.5)^n u[n] - n(0.5)^n u[n] - (-0.5)^n u[n]$$

Find the transfer function $H(z)$.

7.27. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{3z}{(z + 0.5)(z - 0.5)}$$

The output response resulting from the input $x[n] = u[n]$ and initial conditions $y[-1]$ and $y[-2]$ is

$$y[n] = [(0.5)^n - 3(-0.5)^n + 4]u[n]$$

Determine the initial conditions $y[-1]$, $y[-2]$, and the part of the output response due to the initial conditions.

7.28. A linear time-invariant discrete-time system has unit-pulse response $h[n]$ equal to the Fibonacci sequence; that is, $h[0] = 0$, $h[1] = 1$, and $h[n] = h[n-2] + h[n-1]$ for $n \geq 2$. Show that the system's transfer function $H(z)$ is rational in z. Express $H(z)$ as a ratio of polynomials in positive powers of z.

7.29. Consider each of the following transfer functions:

(i) $H(z) = z/(z - 0.5)$
(ii) $H(z) = z/(z + 0.5)$
(iii) $H(z) = z/(z - 1)$
(iv) $H(z) = z/(z + 1)$
(v) $H(z) = z/(z - 2)$
(vi) $H(z) = z/(z + 2)$

(a) Compute the pole of the system. From this pole position, describe the type of behavior that you would expect in the transient response.
(b) Verify your prediction in part (a) by determining an analytical expression for the unit-pulse response.
(c) Simulate the unit-pulse response, and compare with the answer obtained analytically in part (b).

7.30. Consider the following transfer functions:

(i) $H(z) = (z^2 - 0.75z)/(z^2 - 1.5z + 2.25)$
(ii) $H(z) = (z^2 - 0.5z)/(z^2 - z + 1)$
(iii) $H(z) = (z^2 - 0.25z)/(z^2 - 0.5z + 0.25)$

(a) Compute the pole positions. From knowledge of the pole positions, describe the type of behavior you would expect in the transient response.
(b) Without computing the actual response, give a general expression for the step response.
(c) Verify your prediction by simulating the system for a step input.

7.31. By using the z-domain representation, determine the transfer functions of the discrete-time systems shown in Figure P7.31.

7.32. Consider the discrete-time system shown in Figure P7.32.

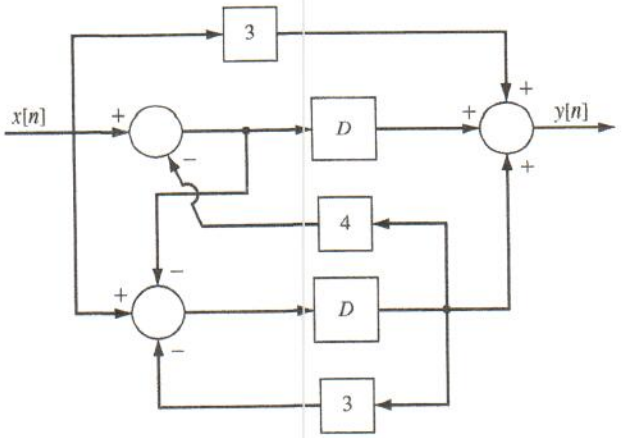
(a) Determine the transfer function $H(z)$ of the system.
(b) Determine the system's input/output difference equation.
(c) Compute the output response $y[n]$ when $x[n] = 4u[n]$ with zero initial conditions.

7.33. Consider the cascade connection shown in Figure P7.33. Determine the unit-pulse response $h_2[n]$ of the system with transfer function $H_2(z)$ so that, when $x[n] = \delta[n]$ with zero initial conditions, the response $y[n]$ is equal to $\delta[n]$.

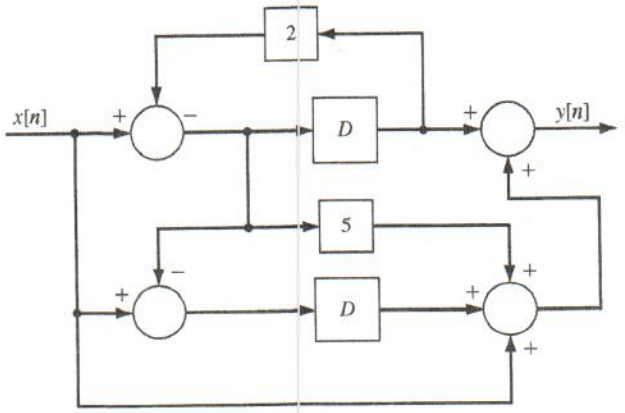
7.34. A linear time-invariant discrete-time system is given by the feedback connection shown in Figure P7.34. In Figure P7.34, $X(z)$ is the z-transform of the system's input $x[n]$, $Y(z)$ is the z-transform of the system's output $y[n]$, and $H_1(z)$, $H_2(z)$ are the transfer functions of the subsystems given by

$$H_1(z) = \frac{z}{z+1}, \qquad H_2(z) = \frac{9}{z-8}$$

(a) Determine the unit-pulse response of the overall system.
(b) Compute the step response of the overall system.
(c) Compute $y[n]$ when $x[n] = (0.5)^n u[n]$ with $y[-1] = -3$, $y[-2] = 4$.
(d) Compute $y[n]$ when $x[n] = (0.5)^n u[n]$ with $y[-2] = 1$, $w[-1] = 2$, where $w[n]$ is the output of the feedback system in Figure P7.34.
(e) Verify the results of parts (a) to (d) via computer simulation.



(a)



(b)

FIGURE P7.31

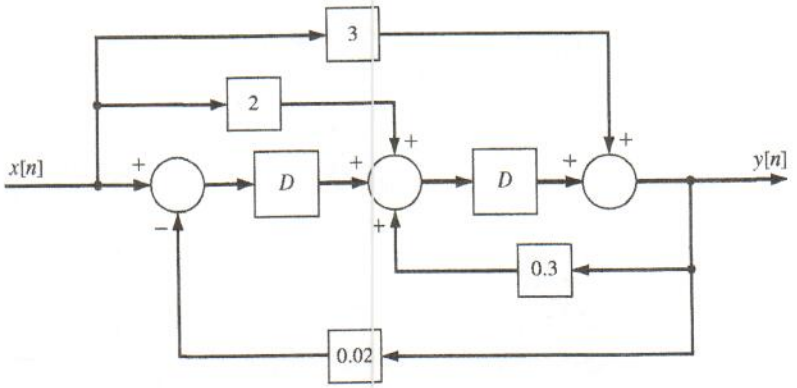


FIGURE P7.32

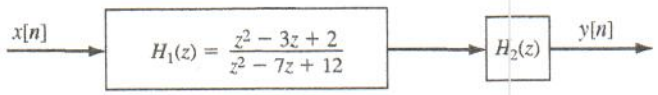


FIGURE P7.33

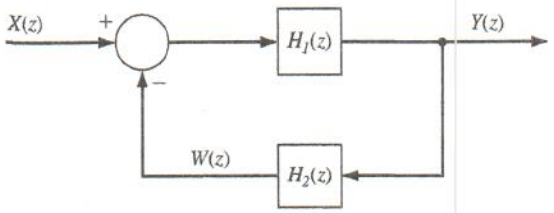


FIGURE P7.34

7.35. A model for the generation of echoes is shown in Figure P7.35. As shown, each successive echo is represented by a delayed and scaled version of the output, which is fed back to the input.

(a) Determine the transfer function $H(z)$ of the echo system.

(b) Suppose that we would like to recover the original signal $x[n]$ from the output $y[n]$ by using a system with transfer function $W(z)$ [and with input $y[n]$ and output $x[n]$]. Determine $W(z)$.

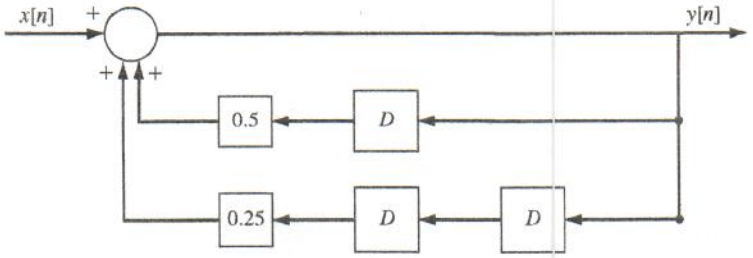


FIGURE P7.35

7.36. A linear time-invariant discrete-time system is given by the cascade connection shown in Figure P7.36.

(a) Compute the unit-pulse response of the overall system.

(b) Compute the input/output difference equation of the overall system.

(c) Compute the step response of the overall system.

(d) Compute $y[n]$ when $x[n] = u[n]$ with $y[-1] = 3, q[-1] = 2$.

(e) Compute $y[n]$ when $x[n] = (0.5)^n u[n]$ with $y[-2] = 2, q[-2] = 3$.

(f) Verify the results of part (a) and parts (c) to (e) via computer simulation.

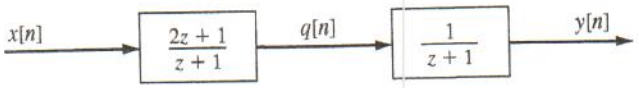


FIGURE P7.36

7.37. A linear time-invariant discrete-time system is excited by the input $x[n] = \delta[n] + 2u[n-1]$. The resulting output response with zero initial conditions is $y[n] = (0.5)^n u[n]$. Determine if the system is stable, marginally stable, or unstable. Justify your answer.

7.38. Determine if the system in Problem 7.32 is stable, marginally stable, or unstable. Justify your answer.

7.39. A discrete-time system is given by the input/output difference equation

$$y[n+2] - y[n+1] + y[n] = x[n+2] - x[n+1]$$

Is the system stable, marginally stable, or unstable? Justify your answer.

7.40. For the following linear time-invariant discrete-time systems with unit-pulse response $h[n]$, determine if the system is BIBO stable.

(a) $h[n] = \sin(\pi n/6)(u[n] - u[n-10])$
(b) $h[n] = (1/n)u[n-1]$
(c) $h[n] = (1/n^2)u[n-1]$
(d) $h[n] = e^{-n}\sin(\pi n/6)u[n]$

7.41. Determine whether or not the following linear time-invariant discrete-time systems are stable:

(a) $H(z) = \dfrac{z-4}{z^2 + 1.5z + 0.5}$

(b) $H(z) = \dfrac{z^2 - 3z + 1}{z^3 + z^2 - 0.5z + 0.5}$

(c) $H(z) = \dfrac{1}{z^3 + 0.5z + 0.1}$

7.42. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{z}{z + 0.5}$$

(a) Find the transient response and steady-state response resulting from the input $x[n] = 5\cos 3n$, $n = 0, 1, 2, \ldots$, with zero initial conditions.
(b) Sketch the frequency response curves.
(c) Plot the actual frequency response curves by using MATLAB.

7.43. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{-az + 1}{z - a}$$

where $-1 < a < 1$.

(a) Compute the transient and steady-state responses when the input $x[n] = \cos(\pi n/2)u[n]$ with zero initial conditions.
(b) When $a = 0.5$, sketch the frequency response curves.
(c) Show that $|H(\Omega)| = C$ for $0 \le \Omega \le 2\pi$, where C is a constant. Derive an expression for C in terms of a. Since $|H(\Omega)|$ is constant, the system is called an *allpass filter.*

7.44. Consider the discrete-time system shown in Figure P7.44. Compute the steady-state output response $y_{ss}[n]$ and the transient output response $y_{tr}[n]$ when $y[-1] = x[-1] = 0$ and $x[n] = 2\cos(\pi n/2)u[n]$.

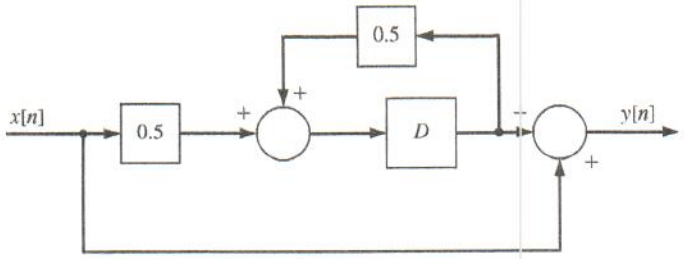


FIGURE P7.44

7.45. A discrete-time system is given by the input/output difference equation

$$y[n+2] + 0.3y[n+1] + 0.02y[n] = x[n+1] + 3x[n]$$

(a) Compute the transient and steady-state output responses when $x[n] = (\cos \pi n)u[n]$ with zero initial conditions.

(b) Use MATLAB to plot the frequency response curves of the system.

7.46. A linear time-invariant discrete-time system has transfer function

$$H(z) = \frac{z}{(z^2 + 0.25)(z - 0.5)^2}$$

(a) Assuming that the initial conditions are zero, find the transient and steady-state response to $x[n] = 12\cos(\pi n/2)u[n]$.

(b) Use MATLAB to plot the frequency response curves of the system.

7.47. The *differencer* is the discrete-time system with the input/output difference equation

$$y[n] = x[n] - x[n-1]$$

Use MATLAB to plot the frequency response curves of the differencer.

7.48. As discussed in Section 7.5, the IIR EWMA filter is given by the transfer function $H(z) = (1 - b)z/(z - b)$, where $0 < b < 1$. Note that the pole of the filter is located at $z = b$, and since b is positive, the pole is on the positive real axis of the complex plane.

(a) Consider the IIR filter with transfer function $H(z) = (1 - b)z/(z + b)$, where $0 < b < 1$. Note that the pole of this system is on the negative real axis of the complex plane. Use MATLAB to plot the frequency response curves of the system when $b = 0.8$. What type of filter is this? Explain.

(b) Now consider the series connection of the IIR EWMA filter and the filter in part (a). The transfer function of the filter is

$$H(z) = \frac{(1 - b)^2 z^2}{(z - b)(z + b)}$$

Use MATLAB to plot the frequency response curves of the system when $b = 0.8$. What type of filter is this? Explain.

7.49. For various values of b in the EMA filter, apply the MACD trading strategy and the modified MACD strategy to the closing price of QQQQ for the period from February 1, 2005 through September 1, 2005. For the values of b that are used, what is the net gain or loss over the time period?